

## Circular Accelerators: Resonance

### Effects in Rings

A subset of this material is

- Overview presented by Reiser in § 3.8.6  
Pgs 166-171.
- Floquet Coordinates
- Perturbed Hill's equation in Floquet Coordinates
- Sources of and forms of Perturbation Terms.
- Unperturbed solution - relation to a simple harmonic oscillator.
- Perturbative analysis of perturbed Hill's equation and resonances
- Tune restrictions resulting from resonances and machine operating points.
- Effects of Space Charge

Overview

In our treatment of single particle orbits of lattices with s-varying focusing we derived transverse particle equations of motion for a particle subject to linear forces:

$$\begin{aligned} x''(s) + R_x(s) x(s) &= 0 \\ y''(s) + R_y(s) y(s) &= 0 \end{aligned}$$

Hill's Equations

Where

$$R_x(s), R_y(s)$$

describe the linear applied focusing fields of the lattice. These terms can also include linear space charge forces associated with the self-fields of a uniform density elliptical modeled by the KV distribution.

In analyzing the Hill's equations we employed phase-amplitude methods:

$$x = A_x^0 W_x(s) \cos \psi_x(s)$$

$$W_x''(s) + R_x(s) W_x(s) - \frac{1}{W_x^3(s)} = 0 \quad ; \quad \psi_x = \psi_{x0} + \int_{s_1}^s \frac{ds'}{W_x^2(s')}$$

$$W_x(s+L_p) = W_x(s)$$

This enabled us to simply identify the Courant-Snyder Invariant

$$\left(\frac{x}{w_x}\right)^2 + \left(\underline{w_x x' - w'_x x}\right)^2 = \text{const}$$

which allowed us to insightfully interpret the dynamics independent of specific particle initial conditions.

We will now show that this formulation can also be exploited to better (analytically?) understand resonant instabilities in periodic focusing lattices.

By choosing coordinates such that stable unperturbed orbits governed by Hill's equation

$$x''(s) + R_x(s)x(s) = 0$$

are mapped to a continuous oscillator:

$$\tilde{x}''(\tilde{s}) + b_{po}^2 \tilde{x}(\tilde{s}) = 0$$

$$b_{po}^2 = \text{const} > 0$$

We will more simply understand the effect of perturbations. (applied field non-linearities, etc.)

$$x''(s) + R_x(s)x(s) = P_x(x, y, s, \vec{\delta})$$

$$y''(s) + R_y(s)y(s) = P_y(x, y, s, \vec{\delta})$$

Perturbations

Possible vector of other coupled variables

We will restrict analysis in this treatment to:

$$\gamma_0 \beta_0 = \text{const} \quad ; \quad \text{No Acceleration}$$

$$\delta = 0 \quad ; \quad \text{No momentum spread}$$

$$\phi = 0 \quad ; \quad \text{Neglect space-charge}$$

In order to more simply illustrate procedures.

Some comments on the effects of space-charge will be added at the end.

In our analysis we will also take the lattice to be periodic with

$$P_x(s + L_p) = P_x(s)$$

$L_p$  = Lattice Period

Ideal linear fields of ring lattice repeat over each element making up the ring.

For a ring we also have the superperiodicity condition

$$P_x(s + C, \vec{x}_1, \vec{\delta}) = P_x(s, \vec{x}_1, \vec{\delta})$$

$$C = \pi \sqrt{L_p}$$

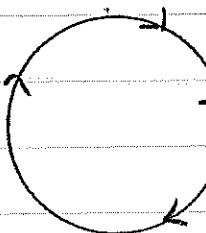
$\sqrt{L_p}$  = "superperiodicity"

This applies to random errors in a ring

random error sources:

Fabrication

Alignment ...

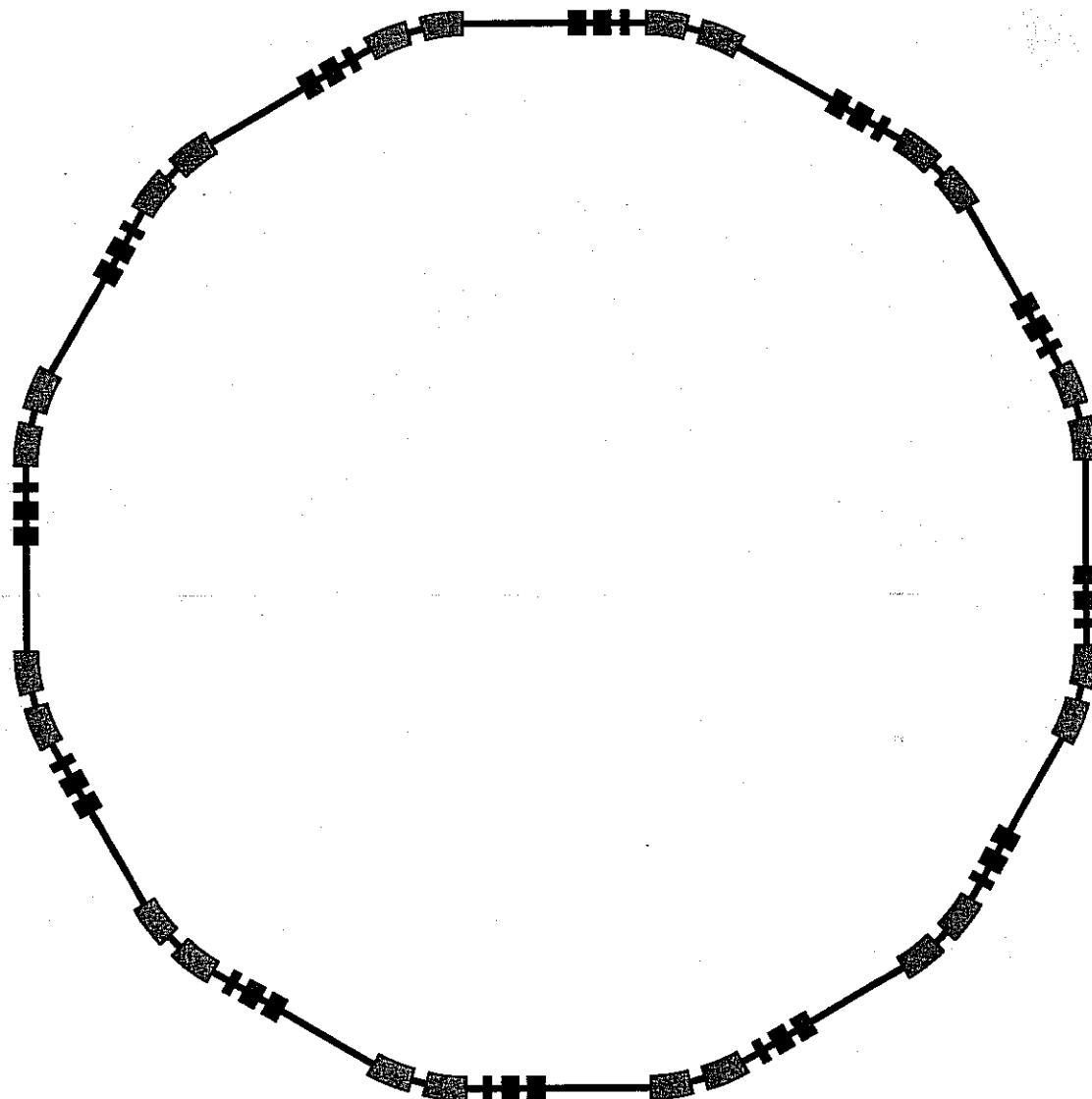


$C$  = Circumference.  
Ring

particle  
encounters same  
error each lap.

SIS – 18 Synchrotron  
GSI, Germany

18 Tesla–Meter Bending Strength

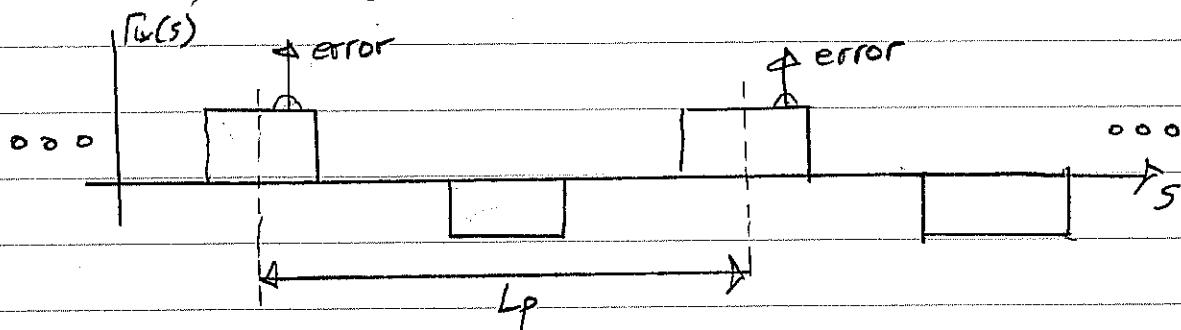


Acceleration Cells  
Not Shown

- Dipole Bending Magnets
- Quadrupole Focusing Magnets (Triplets)

S.M. Lund 6/

This error condition can also apply to systematic errors that occur each lattice period; by taking  $\Delta U = 1$ .



This systematic error condition applies to both ring and linac lattices.

In the following treatments, for simplicity we will (mostly) analyze only the  $x$ -orbit and drop subscript  $x$ 's etc.

## Floquet Coordinates - Connection to a continuous oscillator

Denote: for a stable solution  $x(s)$  to Hill's equation:

"Coord"  
dimensionless

$$U = \frac{x(s)}{\sqrt{B(s)}} \quad B(s) = W(s) = \beta\text{-fron func. of lattice (periodic)}$$

"Angle"  
in ring

$$\varphi = \frac{1}{2\pi} \int_{s_0}^s \frac{ds}{B(\tilde{s})} = \frac{\Delta\Psi(s)}{2\pi}$$

where the tune " $\gamma_0$ " is defined by:

We subscript by zero  
to denote the tune for  
zero self-field.

$$\gamma_0 \equiv \frac{\Delta\Psi(NL_p)}{2\pi} ; \quad N = \text{Superperiodicity} \\ (\text{N=1 for linac})$$

$$\Psi = \text{phase of } x\text{-particle orbit} \\ \Delta\Psi(s) = \Psi(s) - \Psi(s_i)$$

$\gamma_0$  = number of undepressed particle oscillations  
in a ring. (lattice period for linac)

For a linac we take  $N=1$  and  $\gamma_0$  is the fraction of an oscillation that the particle advances  
in a single lattice period.

Now  $\varphi$  can be interpreted as a normalized angle (measured in particle phase advance) of the particle as it advances around a ring. That is,  $\varphi$  advances by  $2\pi$  in one particle transit around the ring.

- In a linac  $\varphi$  advances by  $2\pi$  through a lattice period.

# Perturbed Hill's Equation in Floquet Coordinates

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Take  $\varphi$  to be the independent coordinate and

$$U = U(\varphi)$$

and transform the perturbed Hill's equation

$$X'' + P(s)X = P(x, y, s)$$

to "Floquet" form.

drop additional coupling variables  $\delta$

Here  $P(x, y, s)$  stands for a perturbation

that can represent nonlinear applied fields,

$$U \equiv \frac{X}{\sqrt{\beta'}}$$

$$\Rightarrow X = \sqrt{\beta'} U$$

$$\Rightarrow X' = \frac{\beta' U}{2\sqrt{\beta'}} + \sqrt{\beta'} \frac{dU}{d\varphi} \frac{d\varphi}{ds}$$

Denote

$$\dot{U} = \frac{dU}{d\varphi}$$

$$\bullet = \frac{d}{ds}$$

and note that

$$\frac{d\varphi}{ds} = \frac{1}{2\sqrt{\beta'}}$$

$$\varphi = \frac{1}{2\sqrt{\beta'}} \int_{s_1}^s \frac{ds}{P(\tilde{s})}$$

$$\Rightarrow X' = \frac{\beta' U}{2\sqrt{\beta'}} + \frac{\dot{U}}{2\sqrt{\beta'}}$$

$$X'' = \frac{\beta'' U}{2\sqrt{\beta'}} - \frac{\beta'^2 U}{4\beta'^{3/2}} - \frac{\beta' \dot{U}}{2\sqrt{\beta'} \beta^{3/2}} + \cancel{\frac{\beta' \bullet \dot{U}}{2\sqrt{\beta'} \beta^{3/2}}} + \frac{\ddot{U}}{2\sqrt{\beta'} \beta^{3/2}}$$

Thus

$$x' = \frac{\beta' U}{2\sqrt{\beta'}} + \frac{\ddot{U}}{2\sqrt{\beta'}}$$

$$x'' = \frac{\beta'' U}{2\sqrt{\beta'}} - \frac{\beta'^2 U}{4\beta^{3/2}} + \frac{\ddot{\dot{U}}}{2\sqrt{\beta}^3}$$

and the perturbed Hill's equation

$$x'' + R x = P$$

becomes:

$$\ddot{U} + 2\sqrt{\beta} \left[ \frac{\beta \beta''}{2} - \frac{\beta'^2}{4} + R \beta^2 \right] U = 2\sqrt{\beta}^3 P$$

But

$$\frac{\beta \beta''}{2} - \frac{\beta'^2}{4} + R \beta^2 = 1$$

B-tron amplitude eqn  
 $\beta = w^2$   
 $R = w'' + R w - \frac{1}{w^3} = 0$

by definition, so the term in [ ] vanishes and the perturbed equation reduces to:

$$\ddot{U} + 2\sqrt{\beta}^3 U = 2\sqrt{\beta}^3 P(x, y, s)$$

$$U = U(\varphi) ; \quad \dot{U} = \frac{dU}{d\varphi}$$

$$\varphi = \frac{1}{2\sqrt{\beta}} \int_{s_0}^s \frac{ds}{\sqrt{\beta(s)}}$$

Here we will take in the perturbations:

$$P(x, y, s) \equiv P(\sqrt{\beta} U, y, s(\varphi))$$

Transform other coupled variables similarly.

For  $p=0$  (zero perturbation), the equation is just a simple harmonic oscillator equation with solution:

- Transform has mapped the time dependent unperturbed solution of Hill's equation to that of a simple harmonic oscillator.

$$U(p) = U_0 \cos(\omega_0 p) + \frac{\dot{U}_0}{\omega_0} \sin(\omega_0 p)$$

where we take  $U_0$  and  $\dot{U}_0$  to be the particle initial conditions at  $s=s_0$  with phase choice  $\varphi=0$  at  $s=s_0$  to simplify notation.

The Floquet representation also simplifies the interpretation of the particle dynamics in response to perturbations due to the simple phase-space form of the unperturbed solution!

$$x = \sqrt{\epsilon'} \sqrt{\beta'} \cos \psi \quad ; \quad \psi = \omega_0 p$$

$$U(p) = \frac{x}{\sqrt{\beta'}} = \sqrt{\epsilon'} \cos \omega_0 p$$

$$\dot{U}(p) = \frac{dx}{dp} = -\omega_0 \sqrt{\epsilon'} \sin \omega_0 p$$

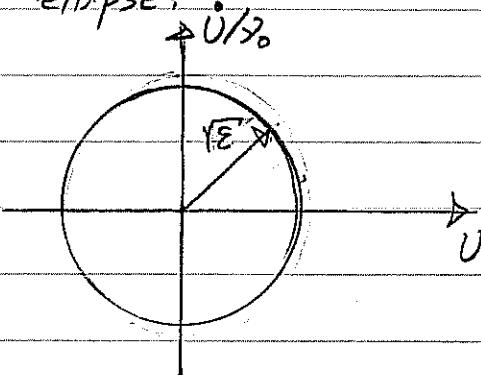
Thus the Courant-Snyder Invariant and the phase space structure are simply expressed in Floquet coordinates.

Courant-Snyder Invariants:

$$U^2 + \left( \frac{\dot{U}}{\omega_0} \right)^2 = \epsilon$$

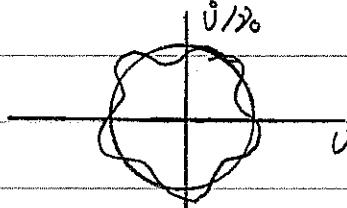
Equation of a circle rather than a rotated ellipse.

Phase Space ellipse:



Phase space is a circle.  
for the unperturbed solution  
to Hill's equation.

Perturbations are more easily understood as distortions  
on circular phase-space.



The  $U - U/\omega_0$  variables also preserve area  
measures of phase space (a feature of the  
transformation being symplectic).

Jacobian

$$\int_{\text{ellipse}} dx \otimes dx' = \int_{\text{circle}} du \otimes d\dot{u} / |J|_{\text{un}} = \int_{\text{circle}} du \otimes \frac{d\dot{u}}{\omega_0} \Big|_{\text{un}}$$

Proof:

$$x = \sqrt{\beta'} u$$

$$x' = \frac{\beta'}{2\sqrt{\beta'}} u + \sqrt{\beta'} \frac{d\phi}{ds} \dot{u} = \frac{\beta' u}{2\sqrt{\beta'}} + \frac{1}{2\sqrt{\beta'}} \dot{u}$$

$$J = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \dot{u}} \\ \frac{\partial x'}{\partial u} & \frac{\partial x'}{\partial \dot{u}} \end{vmatrix} = \begin{vmatrix} \sqrt{\beta'} & 0 \\ \frac{\beta'}{2\sqrt{\beta'}} & \frac{1}{2\sqrt{\beta'}} \end{vmatrix} = \frac{1}{\omega_0}$$

$dx \otimes dx' = du \otimes \frac{d\dot{u}}{\omega_0}$

We now analyze the effects of perturbations on the dynamics

$$\ddot{U} + \omega_0^2 U = \omega_0^2 \beta^{3/2} P(x, y, s)$$

Expand the perturbation in a power series

- Can always be done for all physical applied field perturbations.

$$P(x, y, s) = P_0(y, s) + P_1(y, s)x + P_2(y, s)x^2 + \dots$$

$$= \sum_{n=0}^{\infty} P_n(y, s) x^n$$

and take

$$x = \sqrt{\beta} U$$

to obtain

$$\ddot{U} + \omega_0^2 U = \omega_0^2 \sum_{n=0}^{\infty} \beta^{\frac{n+3}{2}} P_n(y, s) U^n$$

\*  $P$  represents a perturbation due to:

- Systematic or random field errors in magnets etc.

- Alignment error induced field terms etc.

To more simply illustrate resonances we will take the particle to move in the  $x$ -plane only ( $y=0$  for all  $s$ ). If this is not the case the formalism can be generalized by expanding the  $P_n(y, s)$  in a power series in  $y$  and generalizing the notation for the Floquet coordinates to distinguish between the  $x$ - and  $y$ -planes, etc. The essential character of the more general analysis is illustrated by this simple case.

$$y(s) \equiv 0 \quad \text{to simplify picture}$$

In this special case ( $\gamma=0$ ) we expand each coefficient in the power series in a Fourier series as:

Here I implicitly assume a ring and keep  $\psi$  as a  $2\pi$  "phase" path variable in the ring for both systematic and random errors.

$$P_n(\gamma=0, s) \beta^{\frac{n+3}{2}} = \sum_{k=-\infty}^{\infty} C_{n,k} e^{ik\frac{P}{\beta}\psi}$$

$P = \begin{cases} 1 & - \text{A random perturbation (once in ring)} \\ \beta & - \text{A periodic perturbation (every period)} \end{cases}$

$$C_{n,k} = \int_{-\pi/\beta}^{\pi/\beta} d\psi P_n(\gamma=0, s) \beta(s)^{\frac{n+3}{2}}$$

$$s = s(\psi) ; \quad \psi = \int_{s_0}^s \frac{ds}{2\beta(s)}$$

Then the perturbed equation of motion becomes:

$$\ddot{U} + \omega_0^2 U = \omega_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} C_{n,k} e^{ik\frac{P}{\beta}\psi} U^n$$

For the case of small amplitude perturbations this equation can be analyzed perturbatively to linear order as:

$$U = U_0 + \delta U ; \quad |U_0| \gg |\delta U|$$

where:

Simple Harmonic oscillator  $\rightarrow$

$$\ddot{U}_0 + \omega_0^2 U_0 = 0$$

put unperturbed solution in perturbation

Simple  $\rightarrow$  Harmonic oscillator with driving terms.

$$\delta U + \omega_0^2 \delta U \approx \omega_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} C_{n,k} e^{ik\frac{P}{\beta}\psi} U_0^n$$

$U_0$  represents the unperturbed particle orbit.

The general solution to the equation for  $U_0$  can be expressed as:

$$U_0 = U_{0i} \cos(\omega_0 \varphi + \psi_i)$$

$U_{0i}, \psi_i$  initial conditions (constants)

Then,

$$\begin{aligned} U_0^n &= U_{0i}^n \left( \frac{e^{i(\omega_0 \varphi + \psi_i)} - e^{-i(\omega_0 \varphi + \psi_i)}}{2} \right)^n \\ &= \frac{U_{0i}^n}{2^n} \sum_{m=0}^n \binom{n}{m} e^{i(n-m)(\omega_0 \varphi + \psi_i)} e^{-im(\omega_0 \varphi + \psi_i)} \end{aligned}$$

Here,  $\binom{n}{m} = \frac{n!}{m!(n-m)!}$  is a binomial coefficient.

$$= \frac{U_{0i}^n}{2^n} \sum_{m=0}^n \binom{n}{m} e^{i(n-2m)\omega_0 \varphi} e^{i(n-2m)\psi_i}$$

and the perturbed equation becomes:

$$\delta \ddot{U} + \omega_0^2 \delta U \approx \omega_0^2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^n \binom{n}{m} C_{n,k} e^{i(n-2m)\omega_0 \varphi} e^{i(n-2m)\psi_i} \times e$$

In general, can take  $\delta U = \delta U_h + \delta U_p$  where  $\delta \ddot{U}_h + \omega_0^2 \delta U_h = 0$  but in this case we can take  $\delta U_h = 0$  since

this part of the perturbative solution is contained in  $U_0$ . Thus only the particular solution need be found. From the properties of a driven harmonic oscillator, we know that no stable solution exists

whenever the frequency of the driving force equals that of the restoring force (resonant exchange). Thus we have the resonance condition:  $\omega_0 = \pm pk$  see Homework!

$$(n-2m)\omega_0 + pk = \pm \omega_0$$

$$n = 1, 2, 3, \dots ; m = 0, 1, 2, \dots n$$

$$k = -\infty, \dots, 0, \dots \infty$$

$$p = \begin{cases} 1 & \text{A random perturbation (once per ring)} \\ \sqrt{n} & \text{A periodic perturbation (every lattice period)} \end{cases}$$

For  $\omega_0$  satisfying this condition the perturbation will grow in amplitude. If the growth rate is sufficiently large, such tones will be unreliable operating points of the machine and the corresponding perturbation must be corrected. Since this is a linear analysis, the perturbations may be analyzed in turn.

Examples:

$n=0$  (Dipole Perturbation)

$n=0 \Rightarrow m=0$  and the resonance condition becomes:

$$\omega_0 = \pm pk \quad p \cdot k = \text{integer}$$

Therefore:

$$p = \begin{cases} 1 & \text{random pert} \\ \sqrt{n} & \text{periodic perturbation in lattice} \end{cases}$$

$$\omega_0 \neq \text{integer} = \pm pk \quad \text{for dipole } (n=0) \\ = p, p+1, \dots$$

$$p = 1 \quad \text{Note: Random error in ring} \Rightarrow \omega_0 \neq 1, 2, 3, \dots$$

(once per lap)

$$p = \sqrt{n} \quad \text{Systematic error} \\ (\text{every lattice period})$$

$$\omega_0 \neq \sqrt{n}, 2\sqrt{n}, 3\sqrt{n}, \dots$$

Systematic errors less restrictive.

$n=1$  Quadrupole Perturbation

The resonance conditions give:  $\Rightarrow n=1, m=0, \pm 1$

$$\nu_0 + pk = \pm \nu_0 \quad (n=1, m=0)$$

$$-\nu_0 + pk = \pm \nu_0 \quad (n=1, m=1)$$

$pk + \nu_0 = \nu_0$  represents a special case that can be eliminated by "renormalizing" the driving force of the oscillator and  $pk = 2\nu_0$  implies that:

$$p = \begin{cases} 1; & \text{random pert. in ring} \\ 0.5; & \text{periodic pert. in lattice} \end{cases}$$

$\nu_0 \neq \text{half-integer} = |pk/2|$  for quadrupole ( $n=1$ ) perturbations.

 $n=2$  Sextupole Perturbations  $\Rightarrow n=3, m=0, \pm 2$ 

The resonance conditions give:

$$2\nu_0 + pk = \pm \nu_0$$

$$pk = \pm \nu_0$$

$$-2\nu_0 + pk = \pm \nu_0$$

These conditions are equivalent to:

$$p = \begin{cases} 1; & \text{random pert. in ring} \\ 0.5; & \text{periodic pert. in lattice} \end{cases}$$

$\nu_0 \neq \begin{cases} \text{integer (pk)} & \text{for sextupole} \\ \text{half-integer (pk/2)} & (n=3) \text{ perturbations} \\ \text{3rd-integer (pk/3)} \end{cases}$

The integer and half-integer restrictions were already obtained with respect to the dipole and quadrupole cases.

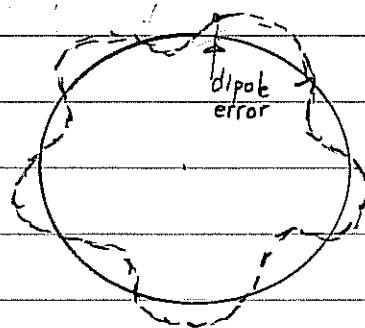
The 3rd integer is a new restriction.

Other cases similar

Aside Interpretation of low-order resonance conditions:

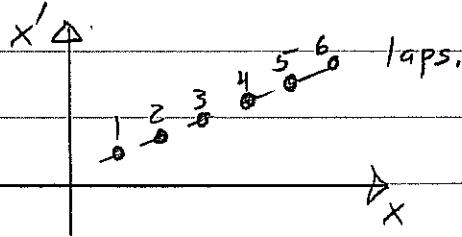
### Dipole Errors

Consider a ring with 1 dipole error along the azimuth of a ring:



an integer tune orbit.

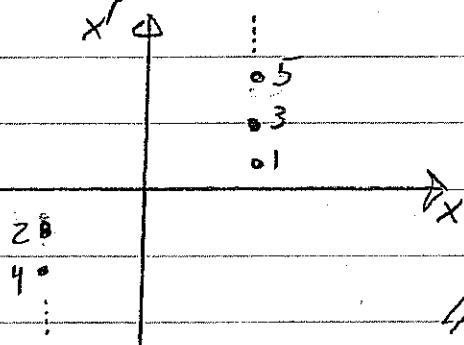
If the particle is oscillating with integer tune, then the particle will experience the error each time with the same phase<sup>of oscillation</sup> and the particle trajectory will "walk-off" lap-to-lap in phase-space. Since the machine aperture is finite the particle will be lost.



### Quadrupole Errors

For a single quadrupole error along the azimuth of a ring, a similar qualitative argument leads one to conclude that if the particle oscillates with  $1/2$  integer tune that the orbit can "walk-off" lap-to-lap.

Phase space patterns as shown here!



The general resonance condition for x-plane motion can be summarized as:

$$M \omega_0 = N$$

M, N integers of the same sign.

M = order resonance

Generally higher order numbers are less dangerous.

Longer coherence path for validity of theory and coefficients generally smaller. Higher order can "wash" out.

In the general case particle motion is not restricted to the x-plane ( $y \neq 0$ ) and a more general resonance analysis shows that:

$$M_x \omega_{0x} + M_y \omega_{0y} = N$$

$\omega_{0x}$  = x-plane tune

$\omega_{0y}$  = y-plane tune

$M_x, M_y, N$  integers of the same sign.

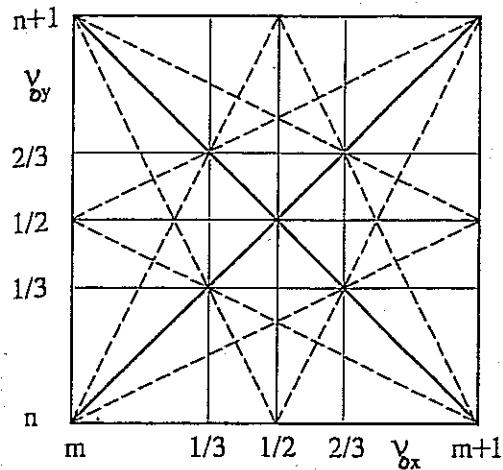
i.e.  $|M_x| + |M_y| = \text{order resonance.}$

These restrictions are plotted order-by-order in  $\omega_x - \omega_y$  plots to find allowed tunes where the machine can safely operate. Generally, lower order resonances are more dangerous, since small effects can invalidate the ideal analysis and "wash out" higher order resonances.

Typical tune plots for up to 3rd order resonances!

$\mathcal{N}=1$  Superperiodicity

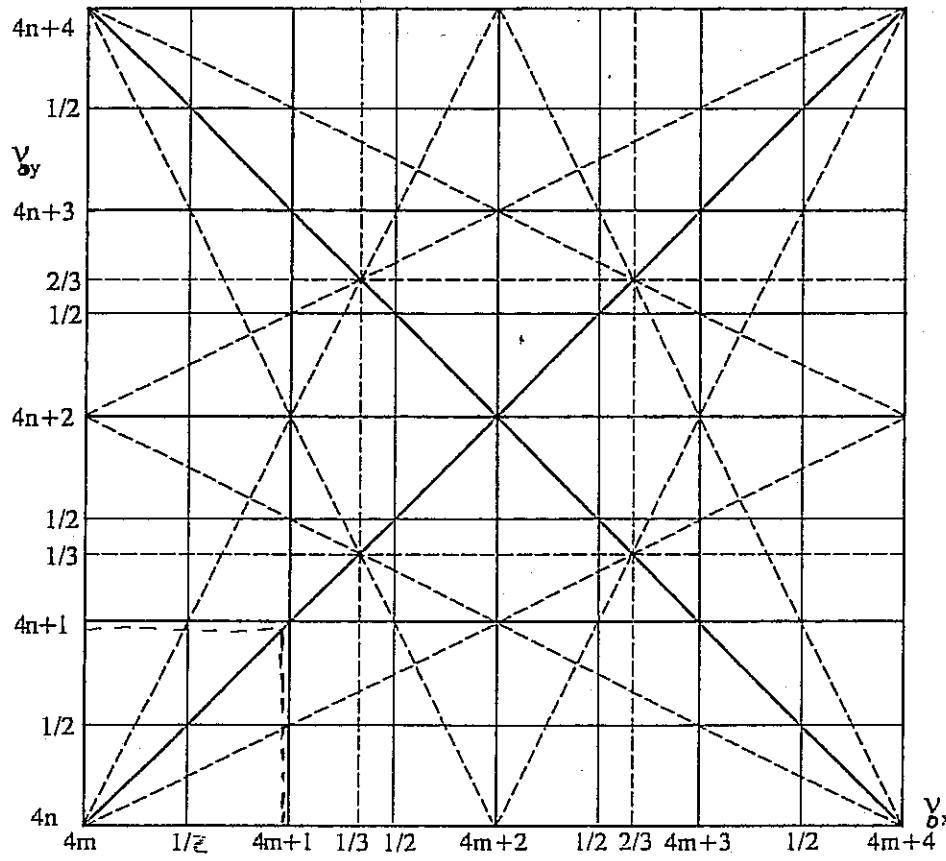
Machine operating point is chosen to avoid resonance lines.



Applicable to field errors from random construction errors,

From Wiedemann

$\mathcal{N}=4$  Superperiodicity



Applicable to systematic lattice field errors in a 4-period ring.

Note lesser density of resonance lines than in the  $\mathcal{N}=1$  case.

Distinguishing between field errors from random construction ( $p=1$ ) and systematic errors on the lattice ( $p=N$ ) is important.

### Random errors ( $p=1$ )

- Errors always present and can give low order resonances.
- Usually have weak amplitude coefficients and may be corrected.

### Systematic errors ( $p=N$ )

- Lead to higher order resonances for large  $N$  and a lower density of resonance lines
  - Large symmetric rings ( $N$  large) have lesser restrictions from systematic errors
  - Practical issues such as construction cost and getting the beam into and out of the ring lead to smaller  $N$
- Amplitude coefficients can be large and the systematic resonances can be strong and thus can be dangerous.

In practice, resonances higher than 3rd order need rarely be considered.

- Effects outside model tend to wash out higher order resonances.

## Effects of Space Charge on Resonances

Machine operating points are generally chosen far from low-order resonance lines. Coherent processes that shift tune values towards a low-order resonance are dangerous.

Coherent - same for each particle in the distribution.

Incoherent - different (random) for each particle in distribution.

Tune shift limits are often called "Laslett Limits" because he <sup>first</sup> calculated such limits for many processes

- image charges
- image currents
- space charge

8  
0

For space-charge, the so-called Laslett limit is taken as:

$$\Delta\gamma = \gamma_0 - \gamma = \Delta\gamma < 1/4$$

$\gamma_0$  = tune at zero space charge  
 $\gamma$  = KV distribution tune

This is probably over idealized and restrictive but is widely used to set ring current limits.

The Laslett condition is probably overly restrictive:

- Real space-charge is not coherent like a kV model
  - spectrum of  $\beta$ -functions,
  - no equilibrium beam and oscillations may crack resonances.
- Simulations indicate that 10's  $\rightarrow$  100's of laps may pose little problems for strong space charge.
  - Univ. Maryland Ring may soon ( $\sim 2007+$ ) operate in such regime to test in the lab.

This could be a good research area with new physics.